Profit and Loss Sharing Contracts and Imperfect Information: A New Dynamic Approach

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Abstract

This paper explores the case of an entrepreneur requiring funds to achieve a project. He acquires financing from a bank, which proposes a profit and loss sharing contract to last throughout the project's lifetime. We analyze the informational asymmetry that benefits the entrepreneur and leads the bank to request participation in the capital of the project. We determine the level of the capital participation based on the costs of liquidation of the project's tangible assets. We then calculate the optimal profit sharing ratio, on the basis of the salary cost opportunity offered to the entrepreneur. We show the asymmetrical character of sharing contracts and calculate the associated default premium. We also show that the ability of each party to break the contract before term, as well as the costs of liquidation constituting the elements of threat/deterrence allowing renegotiation of the original contract's terms, at the end of each exercise, in order to optimize the profit and loss sharing between them. We next establish the conditions under which the project may be carried through to maturity. To our knowledge, this paper develops the first theoretical modeling of a profit sharing ratio for a purely interest-free banking system in a dynamic perspective. Beyond Islamic law's considerations, our paper contributes to formulating an optimal funding contract (by sharing mechanisms).

Key words: Sharing Ratio; Musharaka; Risk; Asymetric Information; Default-Risk Premium.

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Introduction

Contemporary Islamic finance finds its origins in Profit and Loss Sharing (PLS) contracts. Current practice has marginalized this contractual mode, which is essential for consistency and real autonomy of the Islamic financial system, in favor of the other class of contracts authorized by Islamic law, that is to say, the mark-up contracts. Thus, Islamic finance faces a paradox between theory based on PLS and practice dominated by mark-up arrangements.

The literature assumes some responsibility for the weakness of sharing contracts in Islamic financial architecture. Hasan (1985) points out a paradox characteristic of studies during that time: although sharing contracts were a central research issue, significant shortcomings in their modeling remained. Almost thirty years later, the same problems remain. The literature still does not provide an approach for studying these contracts regardless of conventional debt.

In fact, the difficulties experienced by Islamic finance in the use of sharing contracts are due to differences between their former and contemporary uses. Historically, these contracts were designed for commercial activities (Siddiqi, 1991). Today, however, they are used to meet the needs of the company in its current position (especially for industrial and services activities). This involves taking risks of a new nature as well as making changes in the structure of the shared information. Default risk, for example, does not have the same meaning when we compare the former and the present uses of sharing contracts. Information asymmetries between the (contemporary) banks and entrepreneurs are also different from those inherent to the *mudharaba*[2] trade of the Islamic classical era. This is not due to an increase in the risk level but to the emergence of both new risks and asymmetric information related to the nature of contemporary activities. Therefore, this question must be analyzed as a qualitative change in contractual relations and not only as additional risk taking.

We develop in this paper an approach to sharing contracts consistent with their contemporary uses. Our purpose is defining the characteristics of an optimal sharing contract:

- We propose a formulation of the sharing contract which takes into account its information asymmetries:

(i) We determine the optimal level of the entrepreneur’s capital contribution intended to reduce the adverse selection.

[2] Mudharaba is one of the sharing contracts practiced during the classical Islamic age that is rehabilitated (officially at least) by contemporary Islamic banking.

(ii) The capital contribution is fixed in a manner that protects the bank against a premature contractual rupture. At the same time, this eliminates the entrepreneur's incentive to reduce his effort.

- We also calculate the optimal profit sharing ratio. To do this, we rely on the opportunities offered to the entrepreneur as an employee/manager if the sharing contract is not signed. This allows us to obtain a sharing ratio independent of the interest rate.
- We calculate the sharing ratio using a monetary function to describe the entrepreneur preferences. This function is characterized by its translation invariance that perfectly describes the choices of agents subject to interest prohibition. It has been recently developed in the applied mathematical literature and has never been used by Islamic literature.
- We show the asymmetrical character of sharing contracts (different from their inherent information asymmetries). We calculate a quasi-default premium that compensates for the contractual asymmetry of the sharing arrangement.
- We also define the conditions for maintaining the sharing contract from one period to another, its renegotiation or its rupture.

Beyond Islamic law's considerations, our paper is a contribution to formulating an optimal funding contract (by sharing mechanisms).

The rest of the paper is organized as follows. Section 1 reviews the contribution of the literature concerning the sharing contracts and their limits. Section 2 analyzes the informational problems inherent to sharing arrangements. Section 3 is dedicated to the calculation of the optimal profit sharing ratio. Section 4 examines the asymmetric nature of these contracts. Section 5 studies the conditions of a contract's renegotiation and breach. The conclusion presents the main results.

Section 1: The Impossibility of an Autonomous System in the Existing Literature

As one of the most important tenants of Islamic finance, profit and loss sharing contracts (PLSC) have been the subject of several studies. They are modeled, in the existing literature, according to multiple, divergent and sometimes contradictory approaches. In much of this literature, PLSC are backed on conventional loans and therefore take the interest rate as a remuneration reference. However, a part of the literature opts for a stand-alone treatment for these contracts and bases the modeling of their optimal ratio on their own logic.

Nienhaus (1983) represents the first trend in the literature. He suggests adapting the Islamic bank's requirements to those of the conventional loans market. The Islamic bank's profit share should be determined based on the interest rate a conventional bank receives when granting a loan to the same economic agent for the same project. Thus, the sharing ratio fluctuates according to the anticipated profitability of the project, matching the Islamic bank incomes to those of the conventional one. Following this backing of PLSC to interest loans, the author predicts the inability of Islamic banks to compete with their conventional counterparts. He assigns this result to the potential differences between anticipated and actual profits. By forbidding a yield higher than the potential interest, the Nienuhaus' model condemns Islamic banks to underperformance. Paradoxically, the author does not take into account the ability of Islamic banks to manage this type of risk when they are not subject to restrictions similar to those introduced in his model, i.e. alignment with interest rate.

Hasan (1985) focuses on the requirements of the entrepreneur, that is to say the demand side of PLSC contracts. His model aims to explain the inherent characteristics of these contracts and their implications on the sharing ratio, particularly by focusing on what he calls "the fair level of profit sharing." However, this fair level is dependent on determinants of the conventional financing market. The sharing ratio equilibrium, the author finds, is a function of the conventional compartment interest rate, the overall profit rate on investment, the debt level and the risk premium attached to the nature of Islamic contracts. Thus, similar to the Nienhaus' (1983) model, PLSC pricing follows the conventional financing scheme, which prevents highlighting its own logic.

In his later works (2002 and 2010), devoted to amending and extending his initial model of the sharing ratio, Hasan recalls debates on the PLSC he raised in 1985 and develops explanations for their unpopularity. However, he does not propose substantial changes to his original model. He takes for granted the results of his 1985 article and the sharing ratio determinants of his previous model.

Studying some features of the *mudharaba* from the perspective of asymmetric information, Jouaber and Mehri (2012) propose a profit sharing ratio model. In this model, the entrepreneur's choices depend on the interest loans market. As in the two previous models, this modeling scheme submits sharing contracts to the interest loans market.

To model the profit sharing ratio of *mudharaba*, Jouaber and Mehri (2012) analyze the entrepreneur's choice between an Islamic finance sharing contract and a conventional debt one. As in the two previous models, this contribution submits sharing contracts to the conditions of the conventional loan market.

In contrast, a second part of the Islamic literature adopts an autonomous approach to studying *mudharaba*. Its most important contributions are those of Al-Suwailem (2003) and Tag-El-Din (2008).

By focusing on the efforts made by the entrepreneur in a *mudharaba* contract, Al-Suwailem (2003) incorporates the concept of "opportunity cost" faced by the provider of funds. Despite the interest of this contribution, the author does not give an exact definition of the opportunity cost. This opens the way for two interpretations. One could understand opportunity cost as the interest rate offered by conventional banks, which is in contradiction with the author's autonomous approach. It could also be understood as an average yield of similar projects funded through an interest-free system. The second option is no less problematic than the first. Indeed, when the system is purely Islamic, the prior existence of a sharing ratio is a condition for project financing that serves to calculate the average return[3]. Thus, the opportunity cost does not exempt us of determining a base for return on capital (besides interest rate).

Tag-El-Din (2008) also adopts the autonomy scheme in his analyze of PLSC. He builds a set curve of optimal contracts consisting of three alternatives: the *mudharaba* and two other hybrid contracts (part loan and part employee hiring). The first hybrid contract gives the provider of funds a guaranteed income plus a variable part proportional to the project performance. The second one gives the employee a fixed salary and a variable compensation indexed to the project income. The central idea of Tag-El-Din's model is to compare potential differences between the revenue and risk sharing ratios in the three categories of contracts. For each type, a negative relationship is established between each ratio. The higher the income proportion given to a contracting party, the lower is its risk share. The PLS contract (*mudharaba*) is presented, in this model, as a special case of optimal contracts: it is characterized by the equalization of the two ratios (that of income and risk), assuming an identical risk aversion on the part of both contracting parties. As soon as the risk aversions diverge, Tag-El-Din's model predicts a shift of the optimal contract along the optimal contracts curve toward one of the two hybrid categories. Indeed, the results of this paper are questionable. On the one hand, the rational behavior expected of contracting parties is to adjust each's income to the risk that they face (based on their expectations and their risk aversion level). It is irrational that one party would agree to obtain an unchanged or a decreasing income share when her risk share increases. On the other hand, Tag-El-Din's

[3] Which is used directly if these projects are PPP and indirectly for Mark-up contracts.

model opposes the PLS contract (*mudharaba*) to the other contracts. It would be more appropriate to treat them as interchangeable alternatives, as we show in the Section 4.

Thus, according to the existing literature, sharing contracts are difficult to implement in an autonomous perspective. Their remuneration depends either on conventional debt or an opportunity cost difficult to define. One of the main contributions of our paper is overcoming these difficulties by proposing a new scheme of optimal profit and loss sharing, regardless of the conventional credit market.

Furthermore, this paper presents one of the possible ways to formulate an optimal sharing contract. The conventional literature sometimes suggests that sharing contracts are not efficient (Stiglitz and Weiss 1981) or considers them unenforceable because the entrepreneur may not ensure credibly the funds provider that he will not withdraw his human capital before maturity (Hart and Moore 1994). Our objective is to show that it is possible to formulate an optimal sharing contract that overcomes the limits outlined by the conventional literature. In this sense, this paper is also a contribution to the conventional literature, beyond the particularities of Islamic law.

In the following section, we analyze the informational issues inherent to sharing contracts.

Section 2: Informational Issues and Capital Contribution

We consider an entrepreneur with a project[4] expected to be profitable and whose total cost is I. His personal wealth is $w_0 < I$. He solicits external funding from a fund provider, namely a bank. The bank offers him a sharing contract and requires his capital contribution.

Indeed, the capital contribution requirement aims to reduce the information asymmetry between the two parties, especially that concerning the project's quality (Bester 1985 and 1987). Since the direct transfer of information is inefficient because of credibility considerations (Leland and Payle, 1977), the entrepreneur should reveal some of the information he holds in an indirect way. The revelation mechanism must be credible enough to convince the fund provider that his interest is to finance the project, despite the existence of information asymmetry. The contract must be formulated to protect the interests of the fund provider (Stiglitz and Weiss, 1981). In what follows, we analyze the information asymmetries inherent to sharing contracts, relying in particular on the conventional incentive theory (see Appendix 1). We use the entrepreneur's participation in the project capital as an *ex ante* revelation mechanism of the project's quality, determining the level of participation

[4] The funding by PLS contracts concerns production projects of goods or services. This kind of contractual arrangement is not compatible with the funding of goods acquisition, for example.

that gives credibility to the signal emitted by the entrepreneur. We then calculate the participation level that makes the signals credible.

According to the proposed sharing contract, the project's profit is shared between the two parties following a pre-negotiated ratio, and the potential loss incurred is borne according to the parties' participation in the project capital. In Islamic finance, this contract is called *musharaka*.

If the sharing contract is signed, it covers the entire lifetime of the project $[0, T]$, where T is its maturity date. The investment amount I and the sharing ratios α_1 and α_2 are fixed before the implementation of the project, for the entire period. The value of the project's tangible assets[5] is zero at the end of this period. If, however, the entrepreneur does not obtain bank funding, he performs the same functions as an employee/manager.

We assume that the bank finances a long-term project through the sharing contract. The two contracting parties observe the results several times during the project's lifetime and at regular intervals. The contract is signed for the entire project life. However, each party can, after observation of each result, violate its obligations and break the contract. The sharing of profits and potential losses is made when the project results are ascertained.

Moreover, as the entrepreneur's remuneration is entirely dependent on the project's results, and as this is observable only at the end of each period (he works under a veil of ignorance), the entrepreneur is encouraged to maximize his efforts to obtain the best remuneration. Therefore, there is no conflict of interest between the two parties about the entrepreneur's effort. Despite the fact that he is the only one to observe his own effort, the fact that the result is observable at the end of each period prevents the entrepreneur from reducing his effort, as his remuneration depends on it. Furthermore, the entrepreneur cannot get involved in one or more other tasks outside the management of the project financed by the bank. We assume that the project's income will be zero otherwise, which reveals an evident contractual breach, incurring penalties (covering investment losses and the opportunity cost of the capital use).

The conflict of interest between the two parties can occur at the end of each period. Thus, we assume that, by defining the amount of the entrepreneur's participation, the bank aims to protect itself against the entrepreneur's possible withdrawal before the contract's maturity[6] and limit its exposure to this type of loss. If the entrepreneur operates a premature

[5] For example: the machines purchased by the bank and made available to the entrepreneur according to signed PLS contract.

[6] The bank would get a substantial capital participation from the entrepreneur but wants, at the same time, to maximize the chances that he will be able to provide the required amount.

withdrawal, the bank is forced to liquidate assets; it sells them cheaper than their book value (because of the entrepreneur's contractual breach). We assume the following principle: "the party that initiates a premature contractual break assumes the costs of liquidation"[7]. This gives the bank the right to use the entrepreneur's share to cover the difference between the carrying value of its share capital and its liquidation value. Thus, to cover the risk of premature contract breach by the entrepreneur, the bank requires an equity investment that can compensate, at each time t_i, the differential value due to liquidation.

Considering that the project's tangible assets depreciate at a steady cadence until their book value I_{t_i} is zero at $t_i = \mathrm{T}$, we can write:

$$I_{t_i} = \frac{\mathrm{T} - t_i}{\mathrm{T}} \, I_{t_0}$$

Thus, at each time $t_i < \mathrm{T}$, tangible assets have a strictly positive value. However, given the inefficiency of the market's reallocation of tangible assets, it is impossible to sell them at their book value in the case of liquidation. Therefore, the assets' net (market) value is strictly less than their book value. Let L_{t_i} be the liquidation value at time t_i:

$$L_{t_i} < I_{t_i} \qquad \forall \, t_i \in [0, \mathrm{T}[$$

We consider that the assets' liquidation value is known at t_0 and depreciates at the same rate as their book value, that is to say:

$$L_{t_i} = \frac{\mathrm{T} - t_i}{\mathrm{T}} \, L_{t_0}$$

Thus, to protect the bank from any premature and unilateral contractual breach at an entrepreneur's initiative, a capital contribution covering the following difference is required:

$$I_{t_i} - L_{t_i} \qquad \forall \, t_i \in [0, \mathrm{T}[$$ [8]

Since this differential takes its maximum value at the beginning, i.e. t_0[9], the capital contribution required from the entrepreneur is $\vartheta = I_{t_0} - L_{t_0}$.

Considering that $\frac{L_{t0}}{I_{t_0}} = \alpha_1$, the entrepreneur's capital contribution is $(1 - \alpha_1)\, I_0$. Under these conditions, the project is achievable only if the entrepreneur has an initial wealth of:

$$w_0 \geq \left(I_{t_0} - L_{t_0}\right) \qquad (1)$$

The relation (1) is the first feature of the sharing contract that the bank offers to its partner. It means that all candidates with personal wealth less than ϑ are excluded from the financing market. This corresponds to the "eviction" phenomenon (Hart and Moore, 1994). It consists

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[7] In the conventional literature, the liquidation fees should be very significant (see for example Malin and Martimort, 2000).

[8] Even at t_0, i.e. after the conclusion of the contract and the purchase of the tangible assets but before implementation of the project, the assets' liquidation value is lower than their book value.

[9] Given that: $I_t - L_t = \frac{\mathrm{T}-t}{\mathrm{T}} \, (I_0 - L_0)$

in rejecting some viable projects due to the inability of the entrepreneurs to contribute the capital required by the bank.

Note finally that the entrepreneur's willingness to participate in the project's capital means that he accepts to sacrifice diversification through investment in a market portfolio, in favor of investment concentrated in a single project, his project. This willingness expresses the entrepreneur's belief in the good quality of the project and serves as a signal for his fund provider.

In this section we present an important component of the optimal sharing contract under study. We determine the optimal level of the entrepreneur's capital contribution in order to reduce the adverse selection and to eliminate the component of moral hazard linked to the entrepreneur's effort level. In the following sections we continue to analyze other determinants of our optimal sharing contract. Section 3 is devoted to studying the remuneration constraint weighing on the bank; this allows calculation of the optimal profit sharing ratio.

Section 3: Capital Remuneration or Work Force Remuneration

As we have selected a contractual configuration of loss sharing proportional to each party's participation in the project's capital. Then the entrepreneur's profit sharing ratio α_2 is the sole element to be negotiated. The parties rely on exogenous determinants to set this ratio. In the conventional debt contract, the profit sharing takes as reference the lender's remuneration (the interest rate). This is consistent for a contract whose object is financial capital rental. Conversely, a sharing contract combines labor and capital in order to divide the project's profit without rent of finance capital. This contrasts with the explicit or implicit reference to the interest rate made by the literature to calculate the sharing ratio (see Section 1)[10]. This section proposes an alternative based on the absence, in Islamic law, of restrictions on the rental of the work force. Then, the remuneration of a specific category of workers, i.e. the managers, can serve as reference for the profit sharing[11].

Our approach goes beyond the prioritization or exclusion of the different contracts. We consider the salary and sharing contracts as alternative options, having different payment mechanisms but not entirely disconnected. When agents have similar qualifications and perform the same tasks in both contracts, their compensation is supposed to have the same

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[10] Generally, the literature assumes that Islamic banks are in competition with conventional banks, and they should take the interest rate as reference to determine their sharing ratios.

[11] The absence of an anchoring element can be overcome in an exchange economy, given that the contractual arrangements are made separately, depending on specific needs. When the funding must be in an interconnected system, it should be based on shared universal standards.

base. It includes, however, a counterpart for the risk they incur depending on the chosen contract. Based on this understanding, we propose the remuneration of the managerial labor as a landmark for profit sharing and an alternative for predetermined remuneration of financial capital (interest).

We consider that the work necessary for the realization and management of the project is limited to the efforts of the entrepreneur himself[12]. The salary represents, then, an objective constraint that the fund provider must take into account to determine the profit he wishes to obtain from the sharing contract. We consider that a significant number of economic agents have expertise in carrying out certain projects but lack the required financial capital to do it. It is also supposed that there is a significant number of other agents who have the necessary funds to implement similar projects but choose to realize and manage them with employees/managers.

Labor suppliers (managers) do not express the same preferences towards risk. They choose between labor (management) and *musharaka* according to their degree of risk aversion. We can identify an *indifferent agent* whose employee/management guaranteed income (i.e. salary) and uncertain *musharaka* income have the same expected value.

This indifferent agent, derived by ranking labor suppliers according to their risk aversions, divides them into two groups: entrepreneurs and employees/managers. For a given expected net profit, there are a number of agents willing to enter into a *musharaka* contract with the fund providers. The last agent to accept such a contract is the indifferent agent, characterized by the higher risk aversion in the entrepreneurs' population. Other labor suppliers, whose risk aversion is greater than that of the indifferent agent, opt to conclude the salary contract. As wages are exogenous, there is no question of determining one salary for each agent. All agents who choose the salary contract have the same wage, i.e. the average market wage offered for this category of workers/managers.

In the case where the sharing contract proposed by the bank does not meet the expectations of the financing candidate and is therefore not signed, the latter holds the position of manager/employee and obtains the average market salary offered for this type of function $\overline{S}$. Moreover, he places his personal wealth on the financial market, according to Islamic law norms, i.e. an investment with neither predetermined compensation nor guaranteed capital.

[12] If the project requires another work force, its remuneration is considered as a charge.

Thus, the personal wealth of the agent cannot enjoy a safe and remunerated investment. The agent must choose between two different types of risk. He then performs a trade-off in terms of expected utility, between what a market portfolio placement provides him and what he expects to get from investment of the same amount in his own project. We consider, moreover, that the project is fairly small compared to the market (banking) portfolio, and its performance is independent of the latter.

Suppose now that there is a market (banking) portfolio in which the funding candidate invests the amount he could have attributed to his own project[13]. This portfolio consists solely of (positively or negatively) remunerated deposits, with a total value W. The portfolio's gross income is ρ; its net profit is therefore $\rho - W$. The concerned candidate participates in the portfolio with a fraction λ. In the case of success, he enjoys of a net profit $\lambda\,\beta(\rho - W)$, and in the case of failure he assumes a loss of $\lambda\,(\rho - W)$. Thus, his placement's utility in the market portfolio is:

$$U_{E_\lambda}\left[\lambda\,\beta(\rho - W)\,\mathbb{I}_{\{\rho \geq W\}} + \lambda\,(\rho - W)\,\mathbb{I}_{\{\rho < W\}}\right] = W_\lambda \qquad (2)$$

where: β is the proportion of net profit paid to depositors and $1 - \beta$ that returning to the bank.

To conclude the sharing contract, the entrepreneur requires an expected compensation at least equal to the wage he can get plus the revenue of his investment in the market portfolio. The constraint that the bank must meet to conclude the sharing contract with the entrepreneur is then:

$$U_E\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right) \geq \overline{S} + \mathrm{W}_\lambda \text{[14]} \qquad (3)$$

The relation (3) is the second feature of the sharing agreement proposed by the Islamic bank to his partner. It implicitly includes a third feature of this contract: the profit sharing ratio should be strictly higher than the entrepreneur's participation in the capital project (corresponding to his participation in the potential loss) in order to assign an expected remuneration to his work effort:

$$\alpha_2 > \alpha_1 \qquad (4)$$

This condition remains applicable even if the loss sharing ratio is different from that of the participation in the project capital (see Section 4).

[13] We are not interested in the case where the entrepreneur chooses to make a non-contractual risk and therefore not-remunerated investment.

[14] We may write this given the unvariable by translation of the agents' utility in absence of a conventional debt contract.

Proposition 1

The optimal sharing contract proposed by the Islamic bank is characterized by:

- Equity investment condition: $w_0 \geq (I_0 - L_0)$
- Sharing ratios hierarchy : $\alpha_2 > \alpha_1$
- Entrepreneur's remuneration constraint:
$U_E\left[\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right] \geq \overline{S} + W_\lambda$

Application: Translation Invariance of Agents' Choices

We proceed now to a specification of the economic agent's risk preferences under prohibition of *riba*. We choose the entropic function, characterized by translation invariance, to calculate the sharing ratio of *musharaka* (see Appendix 2 for more details about this function).

It takes the following form: $U_E(X) = -\gamma_E \ln \mathbb{E}\left[\exp\left(-\frac{1}{\gamma_E}X\right)\right]$

where γ_E is the entrepreneur's factor of tolerance for the risk.

The entrepreneur's utility obtained from the conclusion of the *musharaka* contract is then:

$$U_E\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)$$

$$= -\gamma_E \ln \mathbb{E}\left[\exp\left(-\frac{1}{\gamma_E}\left(\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right)\right)\right] \qquad (5)$$

Thus: $\alpha_2^* = inf\left\{ \alpha_2 \geq 0 \;\; / -\gamma_E \ln \mathbb{E}\left[\exp\left(-\frac{1}{\gamma_E}\left(\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right)\right)\right] \geq \overline{S} + W_\lambda \right\}$

We assume that the (random) project's relative revenue Z follows the probability distribution of the gamma law. It is shown in Appendix 2 that equation (5) can be rewritten as:

$$-\gamma_E \ln E\left[\exp\left(-\frac{1}{\gamma_E}\left(\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right)\right)\right]$$

$$= -\gamma_E \ln\left(\frac{e^{-\frac{\alpha_1}{\gamma_E}}\left(\frac{\gamma_E\theta}{\gamma_E-\theta\alpha_1}\right)^k}{\Gamma(k)\theta^k}\left(\Gamma(k) - \Gamma\left(\frac{\gamma_E-\theta\alpha_1}{\theta\gamma_E}, k\right)\right) + \frac{e^{\frac{\alpha_2}{\gamma_E}}}{\Gamma(k)\left(\frac{\gamma_E+\theta\alpha_2}{\gamma_E}\right)^k}\Gamma\left(\frac{\gamma_E+\theta\alpha_2}{\theta\gamma_E}, k\right)\right)$$

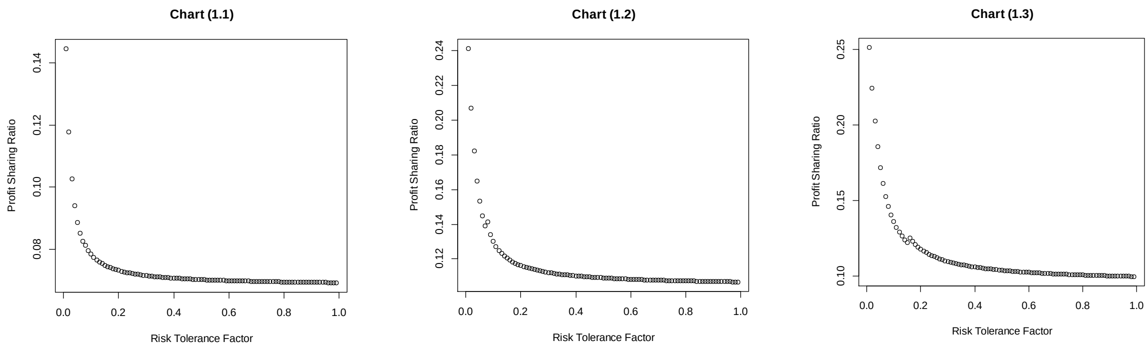
Obtaining an explicit expression of the profit sharing ratio α_2 does not seem possible. Nevertheless, we can calculate this ratio numerically by assigning (ad-hoc) values to the independent variables. Chart (1) represents the values of this ratio according to a set of values of the tolerance factor for the risk $\gamma_E \in [0{,}01; 0{,}99]$ and a loss sharing ratio $\alpha_1 = 0{,}15$.

Chart (1.1) is obtained by setting variable values $\overline{S} = 0.3$ $W_\lambda = 0.075$ $k = 9$ $\theta = 0.5$. For Chart (1.2) we keep the same values of the gamma law $k = 9, \theta = 0{,}5$ and the expected return of the portfolio in which the entrepreneur may place his capital contribution if the sharing contract is not signed $W_\lambda = 0.075$ but we change the salary compensation that the entrepreneur may get as manager if the sharing contract is not signed $\overline{S} = 0.5$. We notice that for each level of risk tolerance, the entrepreneur obtains a higher profit sharing ratio when he has the possibility to get a greater salary.

Chart (1.3) is obtained by retaining the values of the variables in Chart (1.2) and changing the parameters of the gamma distribution $k = 5$ et $\theta = 1$. This corresponds to lower return expectations for the core values and higher for extreme values, compared to Charts (1.1) and (1.2). This results in a decrease in the entrepreneur's share in the expected profit.

Finally, in all scenarios, the more tolerant for risk the entrepreneur is, the lower his requirements in terms of profit sharing ratio.

Chart (1): Profit Sharing Ratio Depending on the Entrepreneur's Risk Tolerance



We conclude this section by emphasizing the fact that the entrepreneur's expected compensation includes a salary equivalent to what he would have received as a manager/employee, in addition to a risk premium that can be positive or negative. This does not mean that income is guaranteed to the entrepreneur. In fact, he has an expected compensation on average (non-linear) equivalent to the salary plus a risk premium that compensates its exposure to the possibility of winning nothing or earning less than the average wage.

Section 4: The Contractual Asymmetry of Sharing Contracts

The previous section shows how the profit sharing ratio is obtained. The objective of this section is to isolate and analyze one component of the sharing ratio. Indeed, sharing contracts are characterized by a contractual asymmetry, different from the information asymmetry analyzed in sections 2 and 3. In most possible contractual configurations, the two contractual parties do not bear the loss in the same proportions. We are interested in the manner in which this asymmetry is taken into account.

We discuss mainly the case where the entrepreneur contributes no funds to the project's capital; this contract is called *mudharaba*. Even if its use is unlikelylowin because of the information asymmetries, the analysis of this contract remains interesting given that it shows the contractual asymmetry of sharing contracts in the best manner. In *mudharaba,* the entrepreneur gets a share α of the profit and the bank a share $1-\alpha$. If the project suffers a loss, the bank supports it entirely (the share of loss is proportional to the capital contribution). Thus, this contract is by definition asymmetric: for the same effort, the entrepreneur obtains a share $\alpha_2$of the positive net profit, while he does not sustain any part of the potential losses. At the same time, the bank receives a proportion $1-\alpha$ of positive net profit and bears the entire loss if the project fails. In other words, while the bank can withstand an opportunity cost (no profit) or a capital loss (negative profit), the entrepreneur is exposed solely to the opportunity cost.

To simplify the calculation, we assume that the entrepreneur is risk neutral; the result does not substantially change if he is risk averse. From equation (3) and knowing that the entrepreneur does not contribute to the project's capital, we can write the constraint that the bank must meet to conclude a *mudharaba* as follows:

$$\alpha\,\mathbb{E}(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} \geq \overline{S} \quad (6)$$

Where: α is the profit sharing ratio (the entrepreneur's share) of *mudharaba*.

If the bank and the entrepreneur sign a symmetrical sharing contract, the previous constraint will be written:

$$\alpha^{*}\mathbb{E}(Z-1) \geq \overline{S} \quad (7)$$

Where: α^{*} is the profit and loss sharing ratio (the entrepreneur's share) of symmetrical sharing contract.

This symmetrical sharing contract assumes that the entrepreneur has provisions (funds not invested in the project capital) enabling him to bear a part of the potential loss proportional to his share in the potential profit. However, the *mudharaba* exempts the entrepreneur from participation in financing losses, limiting his exposition to the opportunity cost (due to the absence of his work remuneration).

When the two constraints are saturated, we have:

$$\begin{cases} \alpha\,\mathbb{E}(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} = \overline{S} \\ \alpha^{*}\mathbb{E}(Z-1) = \overline{S} \end{cases}$$

Consequently:

$$\alpha\,\mathbb{E}(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} = \alpha^{*}\mathbb{E}(Z-1) \quad (8)$$

This relationship expresses the fact that the expected profit for which the entrepreneur is willing to conclude the contract is the same even when the contractual structures are different. Indeed, this simply expresses a different risk sharing from one contract to the other, involving the payment of a risk premium to the party that supports an additional risk. As the bank bears an additional risk in *mudharaba* compared to the symmetrical sharing contract, it necessarily requires additional remuneration.

Relation (8) allows calculating the additional compensation required by the bank for the conclusion of *mudharaba*.

We have: $\alpha^*\mathbb{E}(Z-1) = \alpha\,\mathbb{E}(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}$

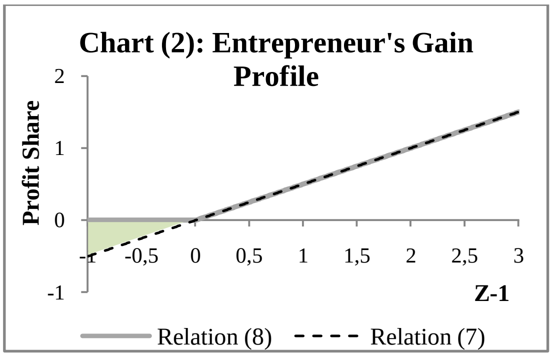
Given that by definition: $\mathbb{E}(Z-1) \leq \mathbb{E}(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}$

Then: $\alpha \leq \alpha^*$

The compensation granted to the bank is consequently:

$$\varphi = \alpha^* - \alpha$$

Thus, to conclude a *mudharaba*, the entrepreneur accepts a smaller sharing ratio α than he could get in a symmetrical contract α^*; the difference between the two represents the premium obtained by the bank in compensation of the *mudharaba's* contractual asymmetry.



The Chart (2)[15] represents the relation (6) corresponding to the entrepreneur's profit profile in *mudhraba* and relation (7) concerning the symmetrical sharing contract. For illustration purposes, we choose the same profit sharing ratio for both contracts. The shaded area shows the losses' share from which the entrepreneur is exempted in *mudharaba* and that the bank assumes fully. To be compensated, the bank should demand a profit sharing ratio $1-\alpha$ that increases its remuneration when the project is profitable. That is to say that α, which is contractually just a profit sharing ratio, becomes economically a profit *and loss* sharing ratio. In other words, this compensation anticipates a possible (non-contractual) failure of the entrepreneur to achieve the level of income expected by the bank ($\pi > I$).

[15] This chart is done according to three assumptions: (i) The entrepreneur is risk neutral (ii) The random variable is defined on the interval [0,4] and (iii) The achievements are equiprobable. The profit sharing ratio is $\alpha = 0{,}5$.

The likeness between this compensation and the default-risk premium of the conventional debt contract is obvious. However, there is a fundamental difference between the two. The definition of default cannot be fully applied to a sharing contract (because the entrepreneur does not agree to pay any amount to the bank, one cannot say that he has defaulted on his contractual commitments when the project makes losses). Thus, we can talk about a *quasi-default premium* integrated into the bank's compensation, in order to correct the contractual asymmetry of *mudharaba*. What distinguishes this quasi-default premium from the classical default-premium is that it is not guaranteed but only expected. It expresses an increase of the bank's share of the expected profit, which by definition is not guaranteed.

When the entrepreneur contributes to the project's capital, i.e. in a *musharaka* contract, the contractual asymmetry is attenuated. Indeed, the bank's exposure to loss is reduced proportional to the entrepreneur's participation in the capital, which necessarily implies a reduction of the required quasi-default premium.

Moreover, one can wonder about the possibility of offering a symmetrical *musharaka* contract, i.e. having the same profit and loss sharing ratio. This issue was a subject of controversy between fiqh schools.

Unity or Duality of the Sharing Ratio

In *mudharaba* the profit sharing ratio is by definition different from the loss sharing ratio. The entrepreneur's share in the loss is zero, while his profit share is positive. However, in *musharaka* the entrepreneur's share in the loss is positive by definition. But what proportion should the entrepreneur bear? Fiqh schools have for a long time differed on this issue. Some schools require the sharing ratio's unity[16] so that the entrepreneur and the fund provider share the profit and the loss in the same proportions. The contemporary Islamic finance has mostly chosen the other option, the differentiation of the two ratios, also called the approach of differing ratios.

Nevertheless, some contemporary contributions (Hasan, 1985) defend the unity of the sharing ratio. This choice does not take into account the contractual asymmetry of sharing contracts (previously analyzed). Indeed, if the fund provider bears a part of the potential losses, he enjoys a larger share of profit than if he assumes no loss. The risk sharing modifications necessarily involve compensation adjustments.

[16] The entrepreneur cannot, for example, get 55% of profit in return for assuming 35% of the losses. This does not mean that the profits and losses are to be shared at 50% for both parties but that the share of each party must be unique, at 60% (loss or profit) for the first party and 40% for the other, for example.

If the bank participates in the project capital with proportion $1 - \alpha_1$ and the entrepreneur with proportion α_1, and if we consider that the entrepreneur's share of profit and loss is also α, the expected profit can then be written:

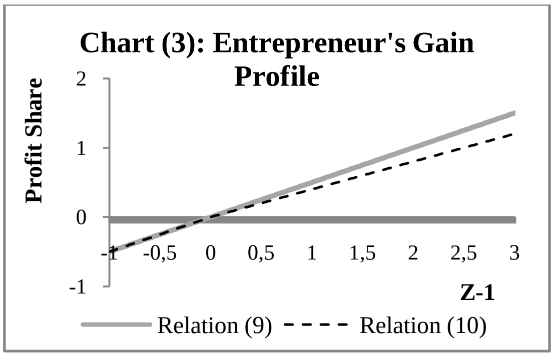
$$G_E^{sy} = \int_0^{\infty} \alpha_1 \, (y-1) \, f_Z(y) dy \qquad (9)$$

Where : f_Z is the density function of the random variable Z and y is the integration variable.

While the contractual asymmetry is at the expense of the fund provider in the *mudharaba*, the entrepreneur bears it in the previous form of *musharaka*, as his work effort is unpaid. It seems possible to solve this problem by providing him a profit share α_2 greater than his (potential) loss share α_1. This last ratio, i.e. α_1, correspond to the entrepreneur's participation to the project's capital. Thus, the expected profit of the entrepreneur is:

$$G_E^{asy} = \int_0^{\infty} \left(\alpha_2 (y-1) \mathbb{I}_{\{y-1 \geq 0\}} + \alpha_1 (y-1) \mathbb{I}_{\{y-1<0\}}\right) \; f_Z(y) dy \qquad \text{with} \;\; \alpha_2 > \alpha_1 \qquad (10)$$

Chart (3)[17] represents the entrepreneur's profit profile in the two previous cases. It provides evidence on the differential compensation of which he is deprived in the first configuration of the *musharaka* (expressed by the area between the two curves).



We note, moreover, that the classical fiqh tradition envisaged a *musharaka* contract with a unique ratio α_3. This is necessarily greater than the entrepreneur's capital participation ratio α_1, in order to remunerate his work effort. It must also be greater than α_2 to compensate for the entrepreneur's additional exposure to loss. Thus:

$$\alpha_3 > \alpha_2 > \alpha_1 \qquad (11)$$

However, this type of contract is impracticable, given the inability of any marginal increase in the entrepreneur's compensation to offset the increased exposure to losses he assumes. We show this formally in the following.

[17] This chart is done according to the three hypotheses of Chart 1. The entrepreneur's share in capital of the project is $\alpha_1 = 0.15$.

In the single-ratio *musharaka* contract, the expected profit of the entrepreneur is given by an equation having the same form as equation (9):

$$G_E^{sy} = \int_0^{\infty} \alpha_3 \,(y-1)\, f_Z(y) dy \qquad (12)$$

To be accepted by the entrepreneur, this kind of contract (equation 11) should give him an expected profit equal to that afforded by a contract with divergent ratios (relation 10). Consequently, the equalization of the expected profit of the two configurations of *musharka* is the necessary condition for the existence of a single *musharaka* contract ratio.

Thus: $$\alpha_3 \,\mathbb{E}(Z-1) = \alpha_2 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}\right] + \alpha_1 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1<0\}}\right]$$

$$\alpha_3 = \frac{\alpha_2 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}\right] + \alpha_1 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1<0\}}\right]}{\mathbb{E}(Z-1)} \qquad (13)$$

Indeed, the single ratio α_3 can exist only in violation of the relation (11). Based on this relation, we can write:

$$\alpha_3 = \frac{\alpha_2 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}\right] + \alpha_1 \mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1<0\}}\right]}{\mathbb{E}(Z-1)} > \alpha_2$$

Then: $$\alpha_2 \frac{\mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}\right] + \frac{\alpha_1}{\alpha_2}\mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1<0\}}\right]}{\mathbb{E}(Z-1)} > \alpha_2$$

$$\mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1\geq 0\}}\right] + \frac{\alpha_1}{\alpha_2}\mathbb{E}\left[(Z-1)\mathbb{I}_{\{Z-1<0\}}\right] > E(Z-1)$$

Consequently: $$\alpha_1 > \alpha_2$$

This result contradicts relation (11), signifying the impossibility of a unique-ratio *musharaka* contract satisfying the entrepreneur's conditions. The only way for the entrepreneur to consider the single-ratio *musharaka* contract is to accept a profit share α_2 (from the differing ratios contract) less than his ratio of participation in the project capital α_1, which is not an option. We can then say that the single-ratio *musharaka* is an unacceptable from the entrepreneur's point of view.

After determining the main characteristics of the optimal sharing, Section 5 analyzes the maintenance, renegotiation or rupture of this contract.

Section 5: Rupture and renegotiation of contracts

Even if the profit sharing contract is signed for the entirety of the lifetime of project $[0, \mathrm{T}]$, one of the parties may, in certain circumstances, decide unilaterally to break it. This questions the revelation principle, given the limited engagement of the agents (Malin and Martimort

2000). A part of conventional literature focuses on this issue[18], with several studies investigating the perspective of long-term contracts, given that their renegotiation, under threat of rupture, seeks to make them optimal. Indeed, the agent's choices during the first period are expected to transmit important information to the principal before the beginning of the second period. This permits the latter to redefine the conditions regarding the contract's continuity or rupture.

These factors are similarly present in the study of profit and loss sharing contracts: the project results for one (or more) period(s) may influence the parties' decisions concerning the continuation of the project. According to the recorded results and their inclination towards renegotiation, the parties make the decision to maintain the project or liquidate it. In this section, we are interested in the conditions necessary for the continuation of the contract.

The parties' engagement in carrying out the project to maturity corresponds, in our model, to the total amortization of assets. In a case where the contract is broken prematurely, the assets have a book value inferior to their net asset value, which engenders a liquidation cost. As described above, the party initiating the rupture assumes this cost.

Indeed, the parties' ability to break the contract before its completion gives each, according to the circumstances, a power of "threat." This power allows an eventual renegotiation of the initial contract terms. As the amount of the investment is determined at the beginning of the period in an irreversible manner[19] and as the contract's duration corresponds to the lifetime of the project's tangible assets, the only element able to be renegotiated is the sharing ratio. According to the registered results in t_i (with $t_i \in [1, T[$), one of the two parties may require renegotiation of the sharing ratio, threatening to put an end to the contract: the entrepreneur threatens to withdraw his human capital and the bank to liquidate the assets. Underperformance and over-performance may both lead to this type of situation.

Underperformance

In the case of underperformance, the bank may wish to break the contract, assuming the tangible asset costs of liquidation[20]. It does not make this decision unless it anticipates a loss linked to the continuation of the project that exceeds the loss due to liquidation. Indeed, upon the signature of the contract, the bank expects returns averaging the income the financing of

[18] For example: Dewatripont (1988 and 1989); Hart and Tirole (1988); Laffont and Tirole (1993).
[19] Given that liquidation has a strictly positive cost.
[20] One may imagine a case where the bank succeeds in locating a buyer for the project. The buyer will not agree to pay a price (significantly) superior to the liquid value of the tangible assets.

other projects. In this way, if the bank liquidates the project in t_i, it may expect to obtain from usage of the recuperated funds a revenue equal to that which it anticipated (in t_0) from the liquidated project for the period $[t_i, \mathrm{T}]$.

In reality, the poor performance of the project over an interval of time $[t_{i-n}, t_i[$ may lead the bank to revise downward its expectations of the project revenue for the remaining period lifetime $[t_i, T]$. Let $G_{\mathrm{B}}(t_i)$ be the anticipated gain of the bank at the moment of the conclusion of the contract for the period $[t_i, T]$:

$$G_{\mathrm{B}}(t_i) = \int_0^{\infty} \left((1-\alpha_2)\left(y_{t_i} - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{y_{t_i} - \frac{T-t_i}{T} \geq 0\right\}} + (1-\alpha_1)\left(y - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{y_{t_i} - \frac{T-t_i}{T} < 0\right\}} \right) f_Z(y)dy$$

And let $G_{\mathrm{B}}^{(1)}(t_i)$ be its anticipated gain for the same period after downward review of the expectations concerning the project's profitability:

$$G_{\mathrm{B}}^{(1)}(t_i) = \int_0^{\infty} \left((1-\alpha_2)\left(y_{t_i}^{(1)} - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{y_{t_i}^{(1)} - \frac{T-t_i}{T} \geq 0\right\}} + (1-\alpha_1)\left(\mathrm{y} - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{y_{t_i}^{(1)} - \frac{T-t_i}{T} < 0\right\}} \right) f_Z(y)dy^{(1)}$$

Furthermore, the overall cost of liquidating the project's assets at the instant t_i is $I_{t_i} - L_{t_i}$. It is supported in its entirety by the bank, which is at the origin of the liquidation. Thus, the bank refrains from breaking the contract as long as the loss it may incur following asset liquidation is superior to the opportunity cost[21] of the project's continuation. Said differently, from the bank's point of view, the project continues to exist, at each instant t_i, as long as:

$$I_{t_i} - L_{t_i} \geq G_{\mathrm{B}}(t_i) - G_{\mathrm{B}}^{(1)}(t_i) \qquad (14)$$

A contract's breach due to underperformance may be likewise produced at the entrepreneur's initiative. This follows a change of his expectations concerning the profitability of his project. If the entrepreneur takes this initiative, he assumes the liquidation costs, which correspond by definition to the residual book value of his participation in the project's capital.

Thus, the entrepreneur may, under certain circumstances, wish to break the contract even if he loses all his wealth invested in the project. In this case, he acts as manager under a salary contract with a guaranteed income. If he anticipates a part of the project revenue to be inferior to his salary income minus the loss of capital, he decides to liquidate the project.

Let $G_{\mathrm{E}}^{(1)}(t_i)$ be the entrepreneur's anticipated profit for the period $[t_i, T]$ after downward revision of his expectations:

$$G_{\mathrm{E}}^{(1)}(t_i) = U_E\left(\alpha_2 \left(Z_{t_i}^{(1)} - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{Z_{t_i}^{(1)} - \frac{T-t_i}{T} \geq 0\right\}} + \alpha_1 \left(Z_{t_i}^{(1)} - \frac{T-t_i}{T}\right) \mathbb{I}_{\left\{Z_{t_i}^{(1)} - \frac{T-t_i}{T} < 0\right\}} \right)$$

[21] We mention opportunity cost because we consider that the income anticipated by the bank at the moment of signing the contract (for the concerned period) corresponds to the average revenue it would obtain if it used the funds to finance other projects.

The entrepreneur prefers to maintain the project as long as his entrepreneurial revenue is superior to his net employment income[22]:

$$G_{\mathrm{E}}^{(1)}(t_i) \geq \frac{T-t_i}{T}\left(\overline{S}-(I_0-L_0)\right) \tag{15}$$

Proposition 2:

A sharing contract signed between an entrepreneur and a bank, according to the terms of proposition 1, for a duration T, may be broken at an intermediate date $t_i \in [0, \mathrm{T}]$ if one of the two following conditions is not satisfied:

$$\begin{cases} I_{t_i} - L_{t_i} \geq G_{\mathrm{B}}(t_i) - G_{\mathrm{B}}^{(1)}(t_i) \\ G_{\mathrm{E}}^{(1)}(t_i) \geq \dfrac{T-t_i}{T}\left(\overline{S}-(I_0-L_0)\right) \end{cases}$$

If the two preceding conditions are violated simultaneously, the project is necessarily liquidated to the charge of the responsible party, or sharing the costs if the two parties agree. However, if underperformance leads to the violation of the only first condition of proposition2, in other words in the case where:

$$\begin{cases} I_{t_i} - L_{t_i} < G_{\mathrm{B}}(t_i) - G_{\mathrm{B}}^{(1)}(t_i) \\ G_{\mathrm{E}}^{(1)}(t_i) \geq \dfrac{T-t_i}{T}\left(\overline{S}-(I_0-L_0)\right) \end{cases}$$

the entrepreneur may accept a renegotiation of the contract concerning the sharing ratio. In this case, the two parties agree on a new profit sharing ratio $\alpha_3 > \alpha_2$ allowing reestablishment of the first condition. We obtain then:

$$I_{t_i} - L_{t_i} > G_{\mathrm{B}}(t_i) - G_{\mathrm{B}}^{(2)}(t_i) \tag{16}$$

where:

$$G_{\mathrm{B}}^{(2)}(t_i) = \int_0^{\infty} \left(\alpha_3\left(y_{t_i}^{(1)} - \frac{T-t_i}{T}\right)\mathbb{I}_{\left\{\mathrm{y}-\frac{T-t_i}{T}\geq 0\right\}} + \alpha_1\left(\mathrm{y}-\frac{T-t_i}{T}\right)\mathbb{I}_{\left\{y_{t_i}^{(1)}-\frac{T-t_i}{T}<0\right\}}\right) f_Z(y)dy^{(1)}$$

However, the entrepreneur will not accept the new ratio α_3 unless it allows him to meet the second condition of proposition 2. We must then have:

$$G_E^{(2)}(t_i) \geq \frac{T-t_i}{T}\left(\overline{S}-(I_0-L_0)\right)$$

where:

$$G_E^{(2)}(t_i) = U_E\left(\alpha_3\left(Z_{t_i}^{(2)} - \frac{T-t_i}{T}\right)\mathbb{I}_{\left\{Z_{t_i}^{(2)}-\frac{T-t_i}{T}\geq 0\right\}} + \alpha_1\left(Z_{t_i}^{(2)} - \frac{T-t_i}{T}\right)\mathbb{I}_{\left\{Z_{t_i}^{(2)}-\frac{T-t_i}{T}<0\right\}}\right)$$

However, it is not imaginable that the bank would accept a renegotiation of the contract forcing it to lower its proportion of anticipated gains following underperformance if the entrepreneur desires liquidation. This case is excluded given that, by preferring liquidation,

[22] Net loss in capital due to liquidation of the project.

the entrepreneur gives a signal of the poor quality of his project that the bank can only accepts.

Corollary 1:
If the bank initiates the project's liquidation following underperformance, it is possible that the entrepreneur may propose renegotiation of the contract allowing the bank to obtain a higher proportion of anticipated gains. This is only possible if the new sharing ratio satisfies the second condition of proposition 2. This supposes that the entrepreneur's expectations, for the period remaining under contract, are more optimistic than those of the bank.

Over-performance

It is equally possible that the project sees over-performance, which leads the two parties to have a rise in their expectations concerning the project revenue for the rest of its lifetime. In this case, the entrepreneur may wish to renegotiate the sharing ratio in order to obtain a larger share of anticipated gains for the contract's remaining periods. We exclude the case in which the will to renegotiate (for a larger share of anticipated gains) comes from the bank because the entrepreneur will refuse, convinced that he may find a source of financing. Furthermore, if the bank liquidates the project, it cannot hope to obtain from use of the recuperated funds a revenue superior to that generated by the initial project, assuming moreover the liquidation costs for its being the cause of the rupture.

Thus, the only imaginable case is that in which the entrepreneur is the cause of the renegotiation, in order to lower the share of the bank's anticipated gains. If the bank refuses and the entrepreneur decides to withdraw, the project will be liquidated at his cost. If, on the contrary, the bank decides to renegotiate, it takes into account the liquidation costs to weigh upon the renegotiation. It accepts a new ratio of gains sharing $\alpha_4 < \alpha_2$ when:

$$G_{\mathrm{B}}^{(3)}(t_i) = G_{\mathrm{B}}(t_i) + \left(I_{t_i} - L_{t_i}\right) \qquad (17)$$

where:

$$G_{\mathrm{B}}^{(3)}(t_i) = \int_0^{\infty} \left(\alpha_4 \left(y_{t_i}^{(3)} - \frac{T - t_i}{T} \right) \mathbb{I}_{\left\{ y_{t_i}^{(3)} - \frac{T-t_i}{T} \geq 0 \right\}} + \alpha_1 \left(y_{t_i}^{(3)} - \frac{T - t_i}{T} \right) \mathbb{I}_{\left\{ y_{t_i}^{(3)} - \frac{T-t_i}{T} < 0 \right\}} \right) f_Z(y) dy^{(3)}$$

The new contract is expected to grant the bank a profit, for the period $[t_i, T]$, that is superior to that it anticipated at the moment of the conclusion of the first contract for the same period.

Corollary 2:
It is possible to adjust the proportion of anticipated remuneration of the parties if the project promises to be more profitable than expected. Renegotiation is necessarily undertaken to the benefit of the entrepreneur but assures the bank an expected gain higher than that expected at the moment of the contract's conclusion.

Conclusion

In this paper, we propose a new approach to the study of profit and loss sharing contracts. We obtain some new results:

- We define an optimal level of entrepreneur participation in the capital project and a profit and loss sharing rhythm that can: (i) reduce the adverse selection and (ii) eliminate the moral hazard related to the entrepreneur's effort.
- We model the optimal profit sharing ratio independently of the interest rate. To our knowledge, this is the first contribution to do this in Islamic literature. We calculate this ratio by characterizing the agents' choices by a monetary function, which is also a novelty in Islamic literature. The advantage of this type of function is that it accurately describes the agents' choices under the prohibition of interest
- We calculate an (expected) default risk premium paid by the entrepreneur to the bank under sharing agreements. Although the default risk definition is not fully applicable to this type of contract, the contractual asymmetry we have shown involves the payment of a risk premium.
- In a dynamic perspective, we determine the conditions of maintaining, renegotiating or breaking sharing contracts.

Beyond the particularities of Islamic law, this paper presents a formulation of an optimal sharing contract and discusses the various issues related to it.

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Appendix 1: Imperfect Information Theory and its Extensions

The Imperfect Information Theory has emerged following the works of Akerlof (1970) and Arrow (1971). It explains the interest in developing "revelation schemes" to limit the profits the information asymmetry yields to the "informed party" (Brousseau and Glachant, 2000). Several contributions have extended these pioneering works, including the incentive theory on which we rely in studying PLS contracts. This theory is interested in the mechanisms encouraging the informed party (the agent) to reveal his private information to the under-informed party (the principal). It offers mainly adverse selection (*ex ante* asymmetry) and moral hazard (*ex post* asymmetry) models.

There are, in the literature, two other traditions that study the informational issues: (i) Incomplete Contracts Theory and (ii) Transaction Costs Theory (Brousseau and Glachant, 2000).

(i) The Incomplete Contracts Theory (ICT) assumes the impossibility of contracting the future behavior of agents, particularly due to the inability of a third party to check (*ex post*) the central interaction variables between the co-contracting parties. It is interested in the influence of institutions on the contracts' structure. It considers that the judge, representing the instance of contract execution in the last resort, is unable to measure some relevant variables like entrepreneur (or employee) effort.

(ii) The New Institutional Theory of Transaction Costs (TTC) extends the limited rationality hypothesis of the juridical institution (established by the Incomplete Contracts Theory) to the contracting parties. Since limited rationality prevents agents from anticipating the optimal coordination arrangements, the obligations are redefined during the execution of the contract (Brousseau and Glachant, 2000).

Our paper studies some issues that can be analyzed relying solely on the assumptions of the Incentive Theory.

Appendix 2: The Entropy Function

The norms of Islamic law do not forbid agents to maximize their utility (contrary to the view defended by part of the Islamic literature (notably Al-Suwailem 2000)). However, the prohibition of *riba* gives their measurement of utility a certain peculiarity. Indeed, the remunerated debt contract (guarantee of the face value and predetermination of the remuneration) is a possible choice that rational agents may resort to. In the case where the proscription of *riba* is applied, this type of choice is excluded, consequently reducing the field of options. We know that an agent's preferences conform to two criteria: his own subjectivity (his degree of aversion) and the objective constraints he faces. In consequence, the proscription of *riba*, which is a legal restraint and therefore objective, reduces the possibility frontier.

Whether entrepreneur or investor, the agent may not expect a guaranteed income in a sharing contract (for example). He must assume part of the project risks. Concretely, when he has an amount of money he is unwilling to risk, he may not earn a gain from it. For this reason, the usual utility functions do not correctly describe his choices. The prohibition of interest incites the agent to associate certainty with the absence of remuneration, from which stems a property qualified in the literature as "invariance by translation". The overall utility an agent obtains from an uncertain investment and a fixed amount, taken together, is equal to its utility obtained from the investment added to the fixed amount. One expresses:

$$U(X+e) = U(X) + e$$

where $U(.)$ is a utility function characterized by the invariance by translation;

X is the amount of uncertain investment;

e is the fixed amount.

The invariance by translation expresses, thus, a disposition to associate to all profit a contractual risk, which coincides perfectly with the consequences of the prohibition of *riba*.

Indeed, there is a category of utility functions called the monetary functions that fulfill three criteria (including that which corresponds to the prohibition of interest): these are progressive, concave and invariable by translation:

(1) Progressive: $U(X) \geq U(Y)$ for all $X \geq Y$

(2) Concave: $U(\lambda X + (1-\lambda)Y) \geq \lambda U(X) + (1-\lambda)U(Y) \quad \lambda \in [0,1]$

(3) Invariable by translation: $U(X+e) = U(X) + e$ (e is a monetary amount and X an uncertain amount).

If the monetary amount e were, on the contrary, deposited in an account with interest, one would have:

$$U(X+e) \geq U(X) + e$$

These monetary functions have been studied by several authors. We mention some references concerning their use:

- Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999). Coherent measures of risk, *Mathematical Finance*, 9:203-228.
- Barrieu, P., El Karoui, N. (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics*, 9:269-298.
- El Karoui, N. and Ravanelli, C. (2009). Cash subadditive risk measures and interest rate ambiguity. Math. Finance, 19:561-590.
- Föllmer, H. and Schied, A. (2004). Stochastic Finance: An introduction in discrete time, second edition, de Gruyter studies in mathematics 27.

- Jouini, E., Schachermayer, W. and Touzi, N. (2008). Optimal risk sharing for law invariant monetary utility functions. Math. Finance, 18:269-292.

We have chosen the entropic function (belonging to the category of monetary utility) to describe the choice of agents constrained by the prohibition of *riba*. This function fulfills the three preceding criteria and takes the following form:

$$U(X) = -\gamma \ln E\left[\exp\left(-\frac{1}{\gamma}X\right)\right]$$

It is a concave function of X and constitutes in this sense a possible criteria for a “rational” agent, and more precisely, a risk-averse agent.

Appendix 3: Profit Sharing Ratio (demonstration)

Let X be a random variable of density f_X :

$$\mathbb{E}[\varphi(X)] = \int_{-\infty}^{+\infty} \varphi(y)\, f_X(y)\, dy$$

If we attribute to the random variable the gamma probability law (of parameters k and θ), can then write :

$$\mathbb{E}\left[e^{a(Z-1)}\right] = \int_0^{+\infty} e^{a\,(y-1)} \frac{y^{k-1} e^{-\frac{y}{\theta}}}{\Gamma(k)\theta^k} dy$$

$$= \frac{e^{-a}}{\Gamma(k)\theta^k} \int_0^{+\infty} e^{a\,y}\; y^{k-1}\, e^{-\frac{y}{\theta}}\, dy$$

$$= \frac{e^{-a}}{\Gamma(k)\theta^k} \int_0^{+\infty} e^{\left(a-\frac{1}{\theta}\right) y}\; y^{k-1}\, dy = (*)$$

With: $\left(a - \frac{1}{\theta}\right) y = -m$

$m = \left(\frac{1}{\theta} - a\right) y = \left(\frac{1 - a\,\theta}{\theta}\right) y$

We have: $\left(a - \frac{1}{\theta}\right) < 0 \Leftrightarrow \left(\frac{a\,\theta - 1}{\theta}\right) < 0 \Leftrightarrow a < \frac{1}{\theta}$

$$(*) = \frac{e^{-a}}{\Gamma(k)\theta^k} \int_0^{+\infty} e^{-m} \frac{\theta^{k-1}}{(1-a\,\theta)^{k-1}} m^{k-1} \frac{\theta}{1-a\,\theta} dm$$

$$= \frac{e^{-a}\,\theta^k}{\Gamma(k)\theta^k (1-a\,\theta)^k} \int_0^{+\infty} e^{-m}\, m^{k-1} dm$$

$$= \frac{e^{-a}}{(1-a\,\theta)^k}$$

Given that :

$$\int_0^{+\infty} e^{-m}\, m^{k-1}\, dm = \Gamma(k)$$

Thus:

$$\mathrm{E}\left[\exp\left(-\frac{1}{\gamma_E}\left(\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right)\right)\right] = \int_0^{+\infty} e^{c(y-1)^+ - b(y-1)^-} f_z(y)\, dy$$

Where: $b = -\frac{\alpha_1}{\gamma_E}$ et $c = -\frac{\alpha_2}{\gamma_E}$

Given that: $(Z-1)^+ = Max(Z-1\,,\ 0) = \begin{cases} 0 & if \quad z \leq 1 \\ z-1 & if \quad z > 1 \end{cases}$

And: $-(Z-1)^- = Min(Z-1\,,\ 0) = \begin{cases} 0 & if \quad z > 1 \\ z-1 & if \quad z \leq 1 \end{cases}$

Consequently :

$$\int_0^{+\infty} e^{c(y-1)^+ - b(y-1)^-} f_z(y)\, dy = \left(\int_0^1 e^{-b(y-1)^-} f_z(y)\, dy = (**)\right) + \left(\int_1^{+\infty} e^{c(y-1)^+} f_z(y)\, dy = (***)\right)$$

We have:

$$(***) = \int_1^{+\infty} e^{c\,(y-1)} \frac{y^{k-1} e^{-\frac{y}{\theta}}}{\Gamma(k)\theta^k} dy = \int_1^{+\infty} e^{-c}\, e^{y\left(c-\frac{1}{\theta}\right)} \frac{y^{k-1}}{\Gamma(k)\theta^k} dy$$

With: $y\left(c - \frac{1}{\theta}\right) = -n \quad \Leftrightarrow \quad n = \frac{1-\theta c}{\theta} y$

$$(***) = \int_{\frac{1-\theta c}{\theta}}^{+\infty} e^{-n} \frac{\theta^{k-1}}{(1-\theta c)^{k-1}} n^{k-1} \frac{\theta}{1-\theta c} dn \ \ e^{-c} \frac{1}{\Gamma(k)\theta^k}$$

And:

$$\Gamma(z,k) = \int_n^{+\infty} y^{k-1} e^{-y} \, dy$$

$$(***) = \frac{e^{-c}}{\Gamma(k)(1-\theta c)^k} \Gamma\left(\frac{1-\theta c}{\theta}, k\right)$$

Morover:

$$(**) = \int_0^1 e^{-b(y-1)^-} f_z(y)\, dy = \int_0^1 e^{-b(y-1)} \frac{y^{k-1} e^{-\frac{y}{\theta}}}{\Gamma(k)\theta^k} dy$$

$$= \frac{e^b}{\Gamma(k)\theta^k} \int_0^1 e^{-y\left(b+\frac{1}{\theta}\right)} y^{k-1}\, dy$$

Let: $u = y(b+\frac{1}{\theta})$ so: $y = u\frac{\theta}{\theta b+1}$

$$(**) = \frac{e^b}{\Gamma(k)\theta^k} \int_0^{\frac{\theta b+1}{\theta}} e^{-u} \left(u\frac{\theta}{\theta b+1}\right)^{k-1} \frac{\theta}{\theta b+1} du$$

$$= \frac{e^b \left(\frac{\theta}{\theta b+1}\right)^k}{\Gamma(k)\theta^k} \int_0^{\frac{\theta b+1}{\theta}} e^{-u}\, u^{k-1}\, du$$

$$= \frac{e^b \left(\frac{\theta}{\theta b+1}\right)^k}{\Gamma(k)\theta^k} \left(\int_0^{+\infty} e^{-u}\, u^{k-1}\, du - \int_{\frac{\theta b+1}{\theta}}^{+\infty} e^{-u}\, u^{k-1}\, du\right)$$

$$(**) = \frac{e^b \left(\frac{\theta}{\theta b+1}\right)^k}{\Gamma(k)\theta^k} \left(\Gamma(k) - \Gamma\left(\frac{\theta b+1}{\theta}, k\right)\right)$$

$$= \frac{e^b}{(\theta b+1)^k} \left(1 - \frac{\Gamma\left(\frac{\theta b+1}{\theta}, k\right)}{\Gamma(k)}\right)$$

Thus:

$$-\gamma_E \ln E\left[\exp\left(-\frac{1}{\gamma_E}\left(\left(\alpha_2(Z-1)\mathbb{I}_{\{Z-1\geq 0\}} + \alpha_1(Z-1)\mathbb{I}_{\{Z-1<0\}}\right)\right)\right)\right]$$

$$= -\gamma_E \ln\left(\frac{e^{-\frac{\alpha_1}{\gamma_E}}\left(\frac{\gamma_E\theta}{\gamma_E-\theta\alpha_1}\right)^k}{\Gamma(k)\theta^k}\left(\Gamma(k) - \Gamma\left(\frac{\gamma_E-\theta\alpha_1}{\theta\gamma_E}, k\right)\right) + \frac{e^{-\frac{\alpha_2}{\gamma_E}}}{\Gamma(k)\left(\frac{\gamma_E+\theta\alpha_2}{\gamma_E}\right)^k}\Gamma\left(\frac{\gamma_E+\theta\alpha_2}{\theta\gamma_E}, k\right)\right)$$