Concept and Mathematics of Islamic Valuation and Financial Engineering

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Starting from the fundamental principles, whether a designed contract implies a clean bay´ rather than ribā is central to our discussion. We argue that ribā and gharar may easily arise through neglect of risk or inappropriate valuation methods for value and risk of assets and financial instruments. This possibility, in turn, strongly necessitates an estimation of expected forward values, market risk, and default risk which is consistent with Islamic principles of avoiding ribā and gharar. Based on consistent estimatation of risk and return the expected costs of risks should be quantified for all parties to the contract, in order to judge, whether a contract is free from usury (ribā) and evitable risk (gharar) and whether it the remaining inevitable risks are distributed fairly between the counter-parties, for example between the investor (rabbu l-māl) and the entrepreneur (muḍārib) within a muḍāraba contract. Therefore, an unbiased quantitative estimation of the risks and returns is desirable so as to put Islamic principles to work.

We argue that Islamic principles, in particular the avoidance of ribā and gharar should be applied to real economic value in the first place, and not a priori to a monetary value in terms of conventional currency. In order to reconcile monetary value with economic value, we propose a reference currency linked to an appropriate commodity basket, reflecting the common economic realities and needs of the respective monetary union. Based on this currency real economic value can be computed in analogy with conventional financial engineering methods.

In order to reflect global economic needs and realities, a global reference currency should be linked to a basket of commodities including in particular the natural resources necessary to ensure both, sustainable survival of mankind and a sustainable living standard above poverty.

Referring to the recent financial crisis of the European Union, we argue that apart from the common economic realities and needs within a given socio-political union, such as the OIC countries, also the also the different realities and needs should be honoured appropriately. We propose a 3-level construction of reference currencies, reflecting the economic realities and needs globally, for each region, and for each country.

We compare conventional financial engineering, based on zero bond numéraires computed from fixed income forward contracts, with Islamic financial engineering based on numéraires computed from bay´u l-salam or/and

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forward contracts on the basis of the reference currencies relevant for the counter-parties of the contract.

We propose that contract valuation and risk management should be performed on the basis of Islamic financial engineering rooted on the reference currencies reflecting the economic realities and needs relevant for the counter parties. Considering the benefits of such a risk management for social economy, particularly when the Organization of Islamic Cooperation (OIC) countries are considered, we argue that an implementation of the described Islamic financial engineering, enables Islamic risk management making transparent expected return and risks, and enables their fair distribution between the counter-parties.

Islamic financial industry following the guideline of such principles of Islamic financial engineering would be able to contribute to sustainable development by (i) more risk-(and-return)-consciousness reflected in the participatory structure of financial contracts, and thus, (ii) encouraging Islamic financial institutions to innovate financial products consistent with real implementation Islamic principles, reflecting the real necessities of modern business and economy.

Introduction

Recent decades have witnessed an explosion in financial innovation and engineering of novel contracts. First of all, from the so-called 'conventional banking system' emerged increasing need to adapt to the challenges from the demand side. Secondly, particularly also within the so-called 'Islamic finance industry', the incentive to innovate around prohibited and disadvantaged transactions has been unfortunately high. A clash between abused financial engineering and Islamic principles seems to prevail, not only since the renaissance of Islamic finance, and not in the Muslim world only. In medieval Europe canonical law prohibited usury, i.e. *ribā*. European merchants however used a combination of contracts called *contractum trinius* to circumvent prohibition of usury. The construction uses put-call parity to synthesize a conventional interest-bearing loan. During the recent renaissance of Islamic finance, this construction prevailed in practices of some so-called 'Islamic' banks, when they include into *murābaḥa* contracts a combination of conditions similar to *contractum trinius*, thus effectively ensuring and concealing a risk-free profit, which might also be called 'regulatory arbitrage'. Exceeding considerably the inflation rate, such a so-called 'risk-free return' is nothing else but usury, i.e. *ribā*. Such combination of contracts in order to circumvent Islamic principles may be considered as product structuring which is a part of conventional financial engineering.

Here and below, we will be concerned however not so much with product structuring but rather with quantitative valuation of financial instruments. Careful computation of expected value, expected return, and expected risks is a prerequisite not only for risk management, but also already for thorough valuation of compliance of contacts with respect to regulatory requirements including compliance with Islamic law. A clean product structuring therefore depends on a serious valuation of value and risk as the more fundamental and more demanding section of financial engineering, which we will deal with subsequently.

In Europe in particular, the growing public debt of several European Union (EU) member countries, has surfaced already in several financial crises, which escalated by salvation programs for financial institutions. According to IMF 2010, the public debt of Italy and

Greece is significantly exceeding annual GDP, while others (like Portugal, Ireland, and Belgium) are following with over 90% of GDP. Continuing conventional public loan policy, it is likely that European public debt will exceed all possible salvation funds in the long run. Comparing 2008 financial crisis of Iceland with 2010 crisis of Greece, some important difference appears: Iceland had its own currency which helped to soften the consequences, but Greece is currently trapped within the EUR monetary union. Recently enterprises and banks already simulate scenarios for the reintroduction of the Drachma[3] in Greece. The point which was missed, when Greece entered 2001 for the second time[4] into a monetary union, is that, a common currency can be sustainable only within a political region which is sufficiently in its economic realities and needs. If some important factor such as economic productivity differs largely between the participating countries, an artificially introduced monetary union creates artificial fluxes and processes which may be in contradiction to the economic reality of some country. This recently became evident with Greece again. The idea that a monetary union of a region like Europe is a priori going to strengthen its political union has proven again to fail, because in the case of Greece, the EUR did not reflect Greek economic reality, with economic mismatch finally resulting in political instability. We will come back to this point below, when we discuss the construction of reference currencies reflecting real value of economy.

On the background of financial crises in Europe, what can be learned from Islamic finance? On the Banque de France conference, March 4, 2011, Kenneth Rogoff of Harvard University commented: *"Western policymakers and economists often portray Islamic financial systems, with their emphasis on shared risk and responsibility in lending, as less efficient than western systems that put no strictures on debt. Yet one can equally argue that Western financial intermediation is far too skewed towards debt, and as a consequence generate many unnecessary risks."*

In the Muslim world, conventional agreed interest loans are commonly objected for the fact that they avoid a sharing participation in the default risk related to the purpose of the loan. For the same reason, transfer of risk by selling it to another party is often objected. Within the Islamic finance community, this sometimes appears to create a far spread impression that a passive attitude towards risk would be more ethical rather than an active management of the risks involved within a project and related contracts. We would like however to argue that both, transparency about all risk involved, and clear agreements how they are to be shared should be part of any contract. Also we believe that for a single agent, entrepreneur or investor, it should be legitimate to use his portfolio of contracts with different counter-parties in order to minimize his own exposures to different types of risks. In fact, avoidance of *gharar* is a fundamental request according to Islamic discourse, eg, numerous *hadiths* forbidding *gharar* sales. Having accepted this, it is becoming clear that, modern tools of risk management and hedging should be applied appropriately, using all the knowledge we currently have, in order to reduce risk exposures. Quantitative evaluation of risks involved therefore is a necessary precondition in order to enable the counterparties to first obtain transparency about the existing risks and, on the basis of this information, to reach a fair agreement about the mutual distribution of risks. As far Islamic finance is concerned, we currently perceive commonly loose interpretation of quantitative evaluation, and last not the

[3] From 1831 to 2001 the Greek currency was the Drachma

[4] From 1869 to 1914 the Drachma was effectively coupled 100% to the *union monétaire latine*

least, avoidance of mathematics as being too sophisticated. However we'd like to remind that striving for knowledge is any Muslim's duty, rather than condemning it plainly as misleading since the mathematics involved appears too complicated for many non-experts.

One historically grown reason for the adverse attitude of many Muslims towards financial mathematics and, in particular, the fair value approach comes from the conventionally common practices of discounting future cash flows with so-called "risk-free" interest rates often derived from forward rates of inter-bank markets and certain government bonds. Indeed these rates are deceptive. First because they are not really "risk-free" as their name suggests. Secondly, their level is considerably higher than sustainable, because they are indeed driven by and related to fixed interest loans. Moreover, the high level of these rates also contributes to push inflation rates up. Such objections against the common discount curves, based on the aforementioned interest rates, are fully legitimate. Below we will argue indeed for a more flexible and adequate construction of discount curves. We will also point out that the mathematical framework of fair value – based on relative prices with respect to some reference asset – does not require at all that the reference asset should be given by an artificial zero-coupon bond linked to "risk-free" discount rates, which, in turn, relate to fixed interest loans and the conventional inter-bank markets for forward rates. The mathematical framework of fair value works perfectly also with a universal commodity-linked currency, or even equity as a reference asset.

Islamic finance for sustainable development

The role of financial institutions is to provide the capital for projects, in particular, for those projects which are useful or even vital for development of societies with sustainable infrastructures. Notwithstanding this fact, many financial institutions (eg, in Europe), which had been shaken by the last financial crisis, have been observed to refuse credits or offer credits only with unbearably high risk premiums. This attitude has severely threatened the existence of in particular local small and medium size family businesses and enterprises, since these suffer from additional discrimination by the traditional rating systems favouring large international players instead. As it is known, however, the economic and innovative power of the society is driven very much by locally rooted small and medium size enterprises, rather than the big transnational players. This is known to hold true in Europe, as well. And it is very likely to hold similarly also for the group of OIC member countries which have for long being striving similarly for their sustainable development, while facing similar challenges along the way.

As a consequence, it is the task of each government to take care that the financial institutions follow up their duty of providing the required capital for small and medium size companies in order for them to remain operational and continue making essential contributions for an innovative and vital economy. Following the financial crisis of 2007, new regulatory framework such as Basel III encourages credit institutes to impose more severe conditions and tougher rating conditions on entrepreneurs for credits. The regulatory requirements now became the pretence of credit institutes for harsher credit conditions, contradicting to generally decreasing interest rates in Europe, particularly in Germany. Government tried to intervene on this with several measures to enable and to push financial institutions to follow up their duty of providing liquidity for enterprises. However, by the time being, Europe's financial industry reacts only very reluctantly. In this aspect, governments of most OIC countries should be in a better position not only because the influence of governments on the domestic financial sector has traditionally been more powerful, but much more also because the basis of understanding between governments and financial industry is derived from the

common rules of *shariah*. In this context, a very different culture of entrepreneurship has shaped the Islamic finance industry with the commonly accepted participatory means of financing, while fixed income products being obsolete.
In the following, we will sketch some essential mathematical aspects of Islamic finance enabling financial engineering similarly easy and rigorous as in the conventional interested-based finance.

The myth of risk-free interest and fixed income

In this section we will argue that the notion of a risk-free fixed income is a myth kept up by the conventional banking sector, rather than an economic reality. Corresponding rates are set mainly by the agreement of an inter-bank market among conventional creditors. To calibrate a current credit contract to such rates may be questionable particularly for the situations where one or more counterparties of the contract have limited or no access to this inter-bank market.

After we have seen that fixed interest rates by themselves are not suitable as a basis for a risk-neutral measure, we conclude that a new basis for such a measure must be sought. This should be done by taking into account the position of the agent in the market, ie, by carefully investigating her exposure to the various risks of the relevant markets and economies involved.
Furthermore credit spreads should be considered more flexible, i.e. not constant, and not necessarily always positive. A negative credit spread for some period would reflect the possibility that the credibility of the considered enterprise is in fact better than that of the reference.
It is one main issue in Islamic finance that a supposed risk-free interest rate should be close to zero. The reference rates of some countries like Switzerland and Japan have already come very close to this since many years. Also in Europe, after the financial crisis of late 2000s, interest rates dropped drastically. This indicates that the economical reality in interest-based financial markets might essentially honour – sooner or later – the fact that there is *no free lunch* in any market and that the assumption of a risk free return is a myth.

Inflation of currencies, time-value, and Time-less VALUE

To the extent that inflation of any currency is inevitable, a time-value which compensates the expected inflation should be disputable. If we decide that full participation in the inflation risk is not bearable to the investor, hence at least a partial compensation of hedging against inflation risk should be admitted. On the other hand, one might argue that investor and entrepreneur perhaps share some common risks, such as the risks of everyday life, and accordingly, the inflation risk – as it is inevitable to both parties – should also be shared among them rather than being put on one party's shoulders.

Real contracts usually involve several cashflows and/or depend on asset values at different times. A fair contract evaluation requires hence the ability to compare cashflows and/or asset values at different times. The nominal value of assets is usually measured in units of a certain currency. However, the real economic value of this currency may changes with time, e.g .due to inflation. For our fair valuation, we are nevertheless interested in the intrinsic real value of the currency, and of our assets. E.g. in the face of inflation, this real value of assets might be measured by inflation-adjusted prices. These are obtained by adjusting future cash flows by discount factors account just for the expected inflation of the currency.

Viewed superficially, discount factors might be objected for introducing undesirable time-value to cash flows. However, if they are chosen just such to compensate for the time-dependency of the nominal value of the currency, they in fact may yield in fact just the desired time-less real value standard. We would like to emphasize that timelessness of value as suggested by Islam has to be requested for the real value rather than the nominal value in the first place.

Islam advocates in fact the use time-independent measures for value. At times of the prophet (pbuh) gold was much more rare than today, since there was no mining industry yet. It was a fairly good inflation-free currency. Its value was stable and hardly to be influenced. Only exceptional political events such as conquests and sieges could trigger sudden regional gains and losses of huge gold treasures, which then could indeed change locally the real value. Except such extremal events, the real value of gold was stable. Today however, developments of mining industry, demands from high technology, and different markets pressure almost continuously on the value of gold in different directions. Although gold might be still more stable than most paper money, its real value has become much more volatile than at times of the prophet (pbuh).

Hence the challenge is to find the reference asset which represents a timeless stable standard of value, similarly as gold in previous times.

THE GOLD OF 21th CENTURY: Back To CoMMODITY-BASED currencY

From very early times of Islam up to the 20th century, the commodities of gold and silver have played an important role in defining the modern currencies. The histories of the GBP and USD in this respect and, in particular, their successive decoupling from their reference commodities simultaneously with their devaluation is described e.g. in El Diwany (2010).

Within the EU, after some period of fixed cross currency rates, the euro was introduced. Less known however is that an early predecessor of the euro was defined already in early 1930s. The universal European currency intended as a "currency for peace" was called "l'Europa" (Le Fédériste (1933)). It was defined as a basket of several valuable commodities. Much later, Lietaer (2001) introduced a global Trade Reference Currency (TRC), dubbed also as "Terra", comprising of a basket of a dozen internationally circulated currencies.

One advantage of the general concept of a basket of commodities underlying to a reference currency is the increased stability. With its currency linked to a basket of commodities and its monetary authority backed 100 per cent by a sufficiently large reserve of these commodities, artificial depreciation (appreciation) of a country's local currency (domestic prices) would come to an end, together with an improvement in its immunity to the associated monetary and fiscal challenges facing its economy.

An inevitable effect of any commodity-based currency is the increase of efforts for production of or mining for the underlying commodities. Taking into account the challenges of 21st century and beyond, previous choices of baskets did not yet account sufficiently for the aspect of sustainability and desirability of the production of the commodities chosen for the basket.

According to Islam, at times of the Prophet Muhammad (*pbuh*), exploration and production of gold was still desirable as the most precious commodity known at that time. It was not yet challenged by an excessive mining industry with all its social and ecological problems. From

our current modern point view, in our opinion, the negative effects related to the intensive industrial production, such as ecological damages and exploitation of workers, cannot be ignored. In essence, what is true for gold and other metals is also true for oil and gas as well as agricultural commodities. For any commodity, consideration should me made whether and to which extend its production is desirable. So from an Islamic perspective, we would like to argue in favour of a historically conscious reading that the "gold" mentioned in the Holy Qur'an should be read just as synonymous for the "most precious commodity". If we ask ourselves – from the perspective of the global challenges ahead of us – what are these "most precious commodities", we might identify such commodities as cleanly produced renewable energy, clean drinking water, and agricultural products produced according to ecological standards.

Euro Lesson: A Multi-Dimensional Currency System Reflecting Diversity and Community Appropriately

The Euro currency was introduced well-intended, with the idea to enhance the political and economical solidarity between the countries involved and the ultimate goal to make them together more competitive in the world markets arena. However, the recent euro crisis has shown that as long as the economic foundations of the constituents are basically too different the hard entry of one or more countries into a currency union may be a problem. If productivity of the countries highly differs, the stronger countries have to aid the weaker ones in order to avoid a potential collapse in the common currency. As a result, the political disputes going along with this process not only put the common currency union at risk, but even worse may damage also the political union, as such concerns have been central to the Euro crisis.

For example, the EC currently is split about whether to introduce Euro bonds or not. The introduction of Euro bonds would be consequent since there exists already a common currency. The opponents however argue in view of the different economic realities within the member countries. The dilemma is that, any currency has to reflect the economic realities, but the EUR member countries now discover that they will not be able to sufficiently converge on their economic realities.

In our opinion, the choices for countries which belong to a certain political entity (eg, the EU) which is either to share a common currency 100 per cent or withdraw from the monetary partnership, are insufficient in number in the face of the current economical and political realities. The currently practiced "0 or 1" hard entries into a currency system apparently ignore the risk that a similar hard exit from it may endanger the whole system. Therefore, we propose a structured system of world currencies, starting from a global reference currency $C_{w.}$ It is defined by a basket of commodities $S_1 \ldots S_n$ whose productions are globally desirable, ie,

$$C_w = \sum_{i=1}^{n} w_i S_i \qquad (1)$$

Similarly, we may consider a basket C_r of commodities, the production of which is desirable just within a certain region, eg, the political entity of the OIC member countries. C_r acts as the currency of this region. $\Delta C_r := C_r - C_w$ accounts for structural differences of the region from the rest of the world. It may also give additional (nonnegative) weight to some of the commodities in C_w, eg, clean drinking water, which is already within the global world basket. It may also contain new commodities specific for the region. Hence

$$\Delta C_r = \sum_{i=1}^{n} w_i^{(r)} S_i + \sum_{i=n+1}^{m} w_i^{(r)} S_i \qquad (2)$$

Finally, we admit an extra basket for individual countries in order to take into account their particular situation, differing from that of its region as well as the world as a whole.

$$\Delta C_c = \sum_{i=1}^{m} w_i^{(c)} S_i + \sum_{i=m+1}^{l} w_i^{(c)} S_i \qquad (3)$$

The currency C_c of any country will be composed from a worldwide defined fraction β_w of the world currency, the regionally defined fraction β_r of its regional commodity basket, and the remaining local country fraction $(1- \beta_w -\beta_r)$ of its local commodity basket, ie,

$$C_c = \beta_w C_w + \beta_r \Delta C_r + (1-\beta_w - \beta_r)\Delta C_c \qquad (4)$$

The fraction β_w should be negotiated on a world conference, common for all regions. It should be as high as possible in order to meet global challenges. However, the case where $\beta_w <1$ is desirable in order to admit the differences according to the particular situation of different regions of the world.

Similarly, the fraction β_r should be negotiated on a regional conference, common for all countries. It should be as high as possible in order to meet common regional challenges. However, $\beta_r <1$ is desirable in order to admit in this case the differences according to the particular situation of the different countries of the region.

By construction, the higher the value of β_w, the more strongly any two world currencies are correlated. The same is true for the currencies of the countries within a region: the higher the value of $\beta_w+\beta_r$, the stronger the correlation between the currencies of any two countries within a region will be. In particular, the negotiable weight β_r may give to a region, eg, Europe, the flexibility to regulate a partial entry into a currency union with some weight β_r realistically reflecting the degree of economical unification.

Fighting usury: The risk-neutral fair value

The value of an asset can equivalently be quoted in the form of a price or in the form of a rate which is relative to a reference price. Hence, a quoted rate per se does not yet imply *ribā* while, vice versa, quoted prices may conceal *ribā*. The form of the quotation, whether it is given price S_t or rate R_t, is irrelevant to the question of usury.
The equivalence relation is

$$\frac{S(t)}{S(0)} = 1 + R(t)\cdot t \qquad (5)$$

When S_t is unknown and subject to risk, it will be stochastic and one has to consider its expected value $E_Q[S_t]$ under a certain risk measure Q. The conventional present value (at time 0) of the stochastic variable S is then given by

$$S(0) = \frac{1}{1+tR(t)} E_Q[S(t)] \qquad (6)$$

The spot rate $R(t)$ may equivalently be converted into a discount factor, corresponding to the initial value $P(0,t)$ of a zero-coupon bond with a nominal value of $P(t,t)=1$ and time-to-maturity t, ie,

$$P(0,t) = \frac{1}{1+tR(t)} \tag{7}$$

The relation (6) above may then be written as

$$\frac{S(0)}{P(0,t)} = \frac{E_Q[S(t)]}{P(t,t)} = E_Q\left[\frac{S(t)}{P(t,t)}\right] \tag{8}$$

which is simply the martingale expression for the relative price of S with respect to the zero-coupon bond $P(\cdot,t)$, whose price is inversely proportional to the spot rate $R(t)$.

In order to determine whether a certain quoted price or rate implies usury, it is rather important to have at hand an independent method to determine a fair price with a risk-neutral position. The expected price should be computed under a measure Q which in particular compensates any risk which is out of the responsibility of the considered counterparty. The computed value is supposed to match as close as possible the realistic economic expectation.

Conventional finance usually assumes depreciation of cash flows, eg, according to $S(t)$, in a certain currency during time. It further assumes that the appropriate way to account for this is to discount with a particular spot rate curve $R(t)$ derived from forward rates $F_i(t) = F(t;T_{i-1},T_i)$ corresponding to quoted interest rates from certain inter-bank markets (e.g. LIBOR).

From an Islamic finance perspective, the questionable point here is the market from which the quotes are taken. The method of calibration of the expected spot value to the forward curve by itself agrees with the principle of fair value.

Even within the still very conventional setting of $R(t)$ and $P(\cdot,t)$ and relations (6) and (8), there is already the flexibility to adjust the rate $R(t)$ (and equivalently the zero-coupon bond) to the any expected time-dependent performance. It is not at all required that it is linked to the conventional inter-bank markets for deposits or swaps.

If it is linked to performance of equity, commodity, or (as we will consider below) to currency, it may be consistent with principles of Islamic finance. Also compensation of inflation of a currency may be considered as legitimate under circumstances that the counterparty cannot bear to be exposed to inflation risk, e.g. because it is too small and does not have the possibility to protect itself. In this case $R(t)$ could be linked to inflation rate.

Concerning relation (2) above, remarkably, in mathematical finance the meaningful quantities are mostly relative. In particular, price dynamics is investigated mainly with using the relative price $\hat{S} := S/S_0$ with respect to the price S_0 of a reference asset. Several desirable properties for the reference asset are usually postulated. Best stability of its value is one of them. The more the price is stable, the lower its volatility, i.e. the less risk of change is connected to the asset price. In the past, the commodity gold was considered as the most stable physical asset with the least volatility of its price.

Note however that, considering "risk-free" zero bonds or rates is unrealistic.[5] Therefore, we propose, instead of the zero-bond linked to a deterministic rate $R(t)$ to consider another reference asset instead, reflecting the stochastic risk. We propose the commodity-linked currency (4) as a natural reference asset for this purpose. With this choice, relation (8) will be replaced by

$$\frac{S(0)}{\mathrm{C}_{\text{country}}(0)} = \mathrm{E}_{\mathrm{Q}}\left[\frac{S(t)}{\mathrm{C}_{\text{country}}(t)}\right] \tag{9}$$

Note that this relation holds, independent of the auxiliary currency units within which S and $\mathrm{C}_{\text{country}}$ are evaluated. This invariance is an advantage of relative prices. In general, the relative price in (8) will be stochastic. However, the relative price of $\mathrm{C}_{\text{country}}$ to itself becomes trivially constant equal to 1.

Conventional Versus Islamic Valuation

Let us give first a simple description of a forward contract and it conventional valuation. A forward contract agrees a fixed price, the fair forward price F to be paid at delivery time T in exchange for an asset S (say a commodity) having present value $S(0)$ and unknown value $S(T)$ at delivery. A fair value forward contract on some traded asset S should always have value 0 at time 0, the time of contraction. Using relative prices w.r.t. zero bonds having price $P(t,T)$, conventional financial engineering demands

$$0 = \frac{S(0)}{P(0,T)} - \frac{F}{P(T,T)} \text{ ,or equivalently } F = \frac{S(0)}{P(0,T)}. \tag{10}$$

Hence $P(0,T)$ acts as a discount factor. Note that (10) implies a nominal value concept with respect to some conventional currency, but tries to compensate this by taking zero bonds as numéraire. Above, (10) is postulated directly, with deterministic $P(0,T)$. Note that, for independent unknown stochastic prices $S(t)$ of the asset and $B(t,T)$ of the zero bond, it holds

$$\mathrm{E}_{\mathrm{P}}\left[\frac{S(t)}{B(t,T)}\right] = \frac{\mathrm{E}_{\mathrm{P}}[S(t)]}{\mathrm{E}_{\mathrm{P}}[B(t,T)]} = \frac{\mathrm{E}_{\mathrm{P}}[S(t)]}{P(t,T)}, \tag{11}$$

with deterministic prices $P(t,T) = \mathrm{E}_{\mathrm{P}}[B(t,T)]$, from which (10) follows immediately, assuming $F = \mathrm{E}_{\mathrm{P}}[S(T)]$. The independency assumption may hold for equity or commodity assets, when it appears plausible that such assets develop essentially independent from inter bank markets determining $B(t,T)$. When asset prices are non-trivially correlated with the stochastic discount factors, the above derivation of (10) has to be replaced by

$$F = \mathrm{E}_{\mathrm{P}}[S(T)] = \mathrm{E}_{\mathrm{P}}\left[\frac{S(T)}{B(T,T)}\right] = \mathrm{E}_{\mathrm{P}}\left[\frac{S(t)}{B(t,T)}\right] = \frac{S(0)}{P(0,T)}, \tag{12}$$

whence again the fair price of the future contract is simply $S(0)/P(0,T)$.

[5] Even a conventional fixed income market in fact is not risk-free, but suffers in particular also from the risk of changing market interest rates.

Let us now consider the valuation of a forward contract within an Islamic financial engineering framework relative to a reference currency C. According to (9) above, we replace (12) by

$$\mathrm{E}_{\mathrm{Q}}\left[\frac{S(T)}{C(T)}\right]=\mathrm{E}_{\mathrm{Q}}\left[\frac{S(t)}{C(t)}\right]=\frac{S(0)}{C(0)}. \tag{13}$$

Unlike (12) where $B(T,T)=1$ by definition of the zero bond, here $C(T)$ is in general still unknown, stochastic. Furthermore the stochastic forward price $S(t)$ and the stochastic currency $C(t)$ are unlikely to become either independent or fully dependent at maturity. Then only value which can be estimated is their relative value. The fair relative value w.r.t. to the given currency is constant. For the future contract hence the fair price should be specified as relative to the reference currency, simply as $S(0)/C(0)$. Remarkably, the fair price is quite analogous to the conventional future price, which also can be viewed as the relative price w.r.t. a zero bond.

Important from the perspective of Islamic principles is that, in the conventional case the zero bond numeraire is a pure interest based asset, which is linked essentially to inter bank markets, while in the Islamic case the numeraire is a currency, which we propose to be linked to a commodity basket supposed to reflect a more appropriate and relevant economic reality.

Now let us consider the special case of full dependence, $S(t)=a\cdot \mathrm{C}(t)$, i.e. the traded asset is proportional to the reference currency (with constant factor a), and hence proportional to the commodity basket defining C. Then (13) yields

$$\mathrm{E}_{\mathrm{Q}}\left[\frac{S(T)}{C(T)}\right]=\frac{S(0)}{C(0)}=a=\frac{\mathrm{E}_{\mathrm{Q}}[S(\mathrm{T})]}{\mathrm{E}_{\mathrm{Q}}[C(\mathrm{T})]}=\frac{F}{\mathrm{E}_{\mathrm{Q}}[\mathrm{C}(\mathrm{T})]}, \tag{14}$$

and therefore $F=a\cdot \mathrm{E}_{\mathrm{P}}[C(T)]$.

Comparing conventional and Islamic approach, a practically important difference is that, the zero bond prices are computed conventionally purely deterministically from available quotes of money market, future, and swap prices, while for the reference currency and the commodity basket their forward prices are currently not easily available from quotes. Their estimation is however possible via calibrated stochastic processes. When the asset *S(T)* is non-trivially correlated with the reference *C(T)* , then stochastic simulation of their relative value may be necessary in order to determine the expected value in (13). This shows the need of an analogous stochastic forward rate model for commodities in analogy to the conventional LIBOR market model.

We finally emphasize that, the numeraires in (12) and (14) are *not* equivalent numeraires, i.e. they are *not* resulting from transformation of each other by the change of numeraire theorem[6]. They are rather the different choices for the primary numeraire, defining the martingale measure P respectively Q, i.e. defining what is meant by fair value.

This fair value according to (14) becomes more than another arbitrary definition exactly when the reference currency C reflects a real value linked to the real economy. Only in this

[6] See eg Brigo & Mercurio 2007

way the Islamic choice (14) becomes really superior to the conventional choice (13). Otherwise, 'value' would remain just a conventional concept. This is the case in the conventional case, since the interbank interest markets behind zerobond curves for conventional currencies do in most cases not really reflect the economic reality to which the counterparties are exposed. This becomes evident particularly in situations, when the finance industry meets some of their home made crises. Then, the conventional zero curves tend to exhibit artefacts from the conventional banking sector. In particular the bias towards short maturities introduces basis spreads into the forward rates traded and quoted in the conventional sector. Since the 2007 financial crisis, it has been realized now within the conventional banking sector that, because of the basis spread included in quoted forward prices, the discount curve which is supposed to yield fair arbitrage-free prices, can no longer be equal to the (inter bank) market forward curve, rather a 2 curve approach becomes necessary[7]. The interpretation is that, the conventional inter bank markets are biased, rather than risk-neutral.

Risk Profiles and Equivalent Martingale Measures

Let us now consider the dynamics of the relative price, $\hat{S}(t) = S(t)/S_0(t)$. Since a future price $S(t)$, $t > 0$, is subject to the risk of change, its expected value $\mathrm{E}_Q\hat{S}(t)$ is of particular importance, since it is the most objective value one can assign to them. The expected value E_Q depends on the measure Q related to the risk of change. Under the assumptions of no arbitrage and a complete market, there exists a unique measure Q with a risk-neutral expectation, i.e.

$$\mathrm{E}_Q\hat{S}(t) = \hat{S}(0) \tag{15}$$

which is called the martingale measure.

However, if the market is not complete, e.g. due to illiquidity, extreme events, or other reasons, the martingale measure Q and the derived arbitrage-free value are no longer unique. In the case of an incomplete market, it makes even more sense to consider also alternative reference assets. In particular, the reference curve $R(t)$ may be chosen according to a particular tailor-made dynamical risk-aversion profile, agreed by the counterparties of a contract. A particular risk aversion profile corresponds to a certain utility function, which in turn may be used to select a particular martingale measure among several equivalent ones. One might even consider time periods within which $R(t)$ becomes negative.

The important issue from the Islamic perspective is the mutual agreement about the risk profile implied by the choice of $R(t)$.

Determining $R(t)$ rather freely according to a mutually agreed risk-profile, rather than determining it from the corresponding reference currency, might be interesting particularly for direct contracts between investor and entrepreneur, where both of them wave developed a common understanding of the risks involved in the project, and correspondingly have consciously both agreed on a particular non-standard form of the $R(t)$ curve, because for the

[7] See eg Bianchetti 2009 for the theoretical background, and Ametrano & Bianchetti 2009 for practial implementation of the 2 curve approach.

project under consideration this choice is more suitable than the plain comparison with the value of the currency.

The last mentioned alternative approach of direct contractual agreement on a risk-profile and corresponding $R(t)$ requires outmost transparency about the risks involved in the contract. Therefore, in cases where there is no detailed understanding and clarity about risks and the counter party attitudes appear to be rather indifferent, then it might be preferable to stick to a standard choice for the reference asset, namely our proposed commodity-linked currency.

Consequences for Islamic Derivatives as Hedging Instruments

We exemplify the consequences of our modified evaluation framework for the design of Islamic derivatives. It is commonly agreed that such derivatives should be permitted only for the purpose of hedging.

E.g. in Jobst 2010 three examples of synthetic instruments from asset-based investment finance have been given. The valuation nevertheless has been based on conventional measure of value using risk-free discount factors. As a consequence, the call-put parity could be used to synthesize a conventional loan.

If the present value would be computed using our proposed reference asset-based currency, the value of a certain amount of this currency would be constant without, any interest. Call-put parity can no longer be used to synthesize a conventional interest-based loan. Hence motivation for such constructions will become void.

Concluding Remarks

We have shown that the mathematical framework of risk-neutral valuation and arbitrage free contract prices may be applied for Islamic finance, similarly as in conventional finance, provided some modification on some conventional inputs. In particular, the conventional discount rates drawn from inter-bank markets forward interest rates have to be replaced.

A rather mild modification for this goal consists in replacing them by performance rates of underlying equity or commodity. Shifting the reference is from fixed income interest to equity or commodity, compatibility with Islamic principles might be achieved.

As a more challenging method, we propose to replace the conventional interest-based zero rate curve by an alternative reference asset. We propose in particular a commodity-linked multi-dimensional reference currency system. Globally negotiated fraction β_w of the world part, and regionally agreed fraction β_r may provide easy political instruments to regulate the degree of required homogeneity of currencies globally or with in a defined region such as Europe respectively.

The most challenging approach is nevertheless the one which leaves also the most contractual freedom and responsibility at the same time to the contracting counterparties. Provided all counterparties have both, the necessary capability for risk analysis and commitment for risk transparency, then they might agree on a particular zero rate curve freely, according to their commonly agreed risk aversion profile. The zero rates in this case may be linked closely to their common judgement of risk for the project underlying to the contract. The zero rates in this case may be even negative, meaning that a later cash flow is strategically preferable to an earlier one.

References

Ametrano, F.M., Bianchetti, M., 2009, *Bootstrapping the Illiquidity: Multiple Yield Curves Construction for Market Coherent Forward Rate Estimation*. SSRN Working Paper.

Anonymous, 1933, L'Europa – monnaie de la paix, *Le Fédériste*, 1 Jan. 1933

Al-Bashir, M., Al-Amine, M., 2005, *Commodity Derivatives: An Islamic analysis*, pp. 58-98 in: Iqbal, M., Khan, T., 2005, (eds.) Financial Engineering and Islamic contracts, Palgrave MacMillan, New York.

Brigo D., Mercurio, F., 2007, *Interest Rate Models – Theory and Practice*, 2nd edition, Springer Verlag, Heidelberg.

Bianchetti, M., 2009, *Two curves, one price - pricing and hedging interest rate derivatives using different yield curves for discounting and forwarding.* SSRN Working Paper.

El Diwany, T., 2010, *The Problem with Interest*, 3rd edn. Kreatoc Ltd., London.

Hamzawy, A. 2003 *Hatar al-'Aulama: Die Arabische Globalisierungsdebatte – Eine Neuauflage der Kontroverse über Moderne und Authentizität ?* Welt des Islams 43 (2), 173-213.

IMF, 2010, Data and Statistics, *World Economic Outlook Data Base*, July 2010, http://www.imf.org/external/pubs/ft/weo/2010

Jobst, A., 2010, *Derivatives in Islamic Finance*, Paper presented at SSRN Preprint.

Latif, A., M., Janahi, A. Rahim, 2005, *Sharī´ah Alternatives to Government Bonds*, pp. 99-122 in: Iqbal, M., Khan, T., 2005, (eds.) Financial Engineering and Islamic contracts, Palgrave MacMillan, New York.

Lietaer, B., 2001, *The Future of Money*, Random House, London.

Qutb, Sayyed, 1949, *Al-Adala al-Ijtima'iyya fi'l-Islam*, transl. by Hardie, J.B. 1980 *Social Justice in Islam*, Octagon Press, N.Y.

Rogoff, K., 2011, *Summary of Comments*, Banque de France conference, March 4.

Schacht, J., 1982, *An Introduction to Islamic Law*, Clarendon Press, Oxford.